THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Solution to Quiz 1

1. (a) True. If there exists a line which contains all the points, then there exist no three noncollinear points which violates axiom **I3**.

   (b) True. By axiom **I3**, there exist three noncollinear points P, Q and R. By axiom **I1**, there exist unique lines l_{PQ}, l_{QR} and l_{RP} where each of the line passes through the indicated two points. Since P, Q and R are noncollinear, l_{PQ}, l_{QR} and l_{RP} must be distinct. It shows that an incidence geometry contains at least 3 lines.

   (c) False. Note that $\mathbb{R}^2$ with lines in usual sense is an incidence geometry. However, consider $x = 1$ and $x = 2$ which are lines in $\mathbb{R}^2$, they do not have any intersection point.

2. (a) Let m and n be lines. If $m \neq n$, then m is parallel to n if $m \cap n$ is an empty set. If $m = n$, m is parallel to itself.

   (b) Consider $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and $\mathcal{L} = \{l_{ij} = \{i, j\} : 1 \leq i < j \leq 5\}$. Then l_{12} is parallel to l_{45} and l_{45} is parallel to l_{23}, however l_{12} is not parallel to l_{23}. It shows that transitivity does not hold for parallelism and so parallelism does not give an equivalence relation on $\mathcal{L}$.

3. (a) (i) Let $a \in \mathbb{R}$, since $a - a = 0 \in \mathbb{Z}$, so $a \sim a$.

   (ii) Let $a, b \in \mathbb{R}$ and $a \sim b$. Then $b - a \in \mathbb{Z}$, which implies that $a - b = -(b - a) \in \mathbb{Z}$ and so $b \sim a$.

   (iii) Let $a, b, c \in \mathbb{R}$ such that $a \sim b$ and $b \sim c$. Then $b - a, c - b \in \mathbb{Z}$. Therefore, $c - a = (b - a) + (c - b) \in \mathbb{Z}$. Hence, $a \sim c$.

   Therefore, $\sim$ is an equivalence relation on $\mathbb{R}^2$.

   (b) $\mathbb{R}/\sim = \{[(x, y)] : 0 \leq x < 1\}$.

4. (a) Suppose the contrary, there exists a line l which contains all points. Then, there exist no three noncollinear points which violates axiom **I3**. Therefore, there exists at least one point P that does not lie on l.

   (b) Let l be a line.

   By axiom **I2**, there are two distinct points B_1 and B_2 on l.

   By axiom **B2**, there exists B_3 on l such that $B_1 * B_2 * B_3$. By using axiom **B3** repeatly, we show that there is an infinite sequence of points B_n so that $B_n * B_{n+1} * B_{n+2}$ for all natural numbers n.

   (c) We claim that there exists a line l which does not contain A.

   By axiom **I3**, there exist three noncollinear points R, S and T.

   (Case 1) $A \in \{R, S, T\}$

   Without loss of generality, let $R = A$.

   By axiom **I1**, there exists unique line l_{ST} such that $S, T \in l_{ST}$.

   Note that l_{ST} does not contain A, otherwise it contradicts to the assumption that A, S and T are noncollinear.

(Case 2) $A \notin \{R, S, T\}$

By axiom **I1**, there exists unique lines l_{ST} such that $S, T \in l_{ST}$.

If A does not lie on l_{ST}, then l_{ST} is the line required.

If $A \in l_{ST}$. By axiom **I1**, there exists unique line l_{RS} such that $R, S \in l_{RS}$.

If A lies on l_{RS}, then both A and S lie on l_{ST} and l_{AS}. By axiom **I1**, $l_{ST} = l_{AS}$ which is a line that contains R, S and T (Contradiction).

Therefore, A does not lie on l_{ST}

Then by (b), there are infinitely many points on l. For each point $X \in l$, by axiom **I1**, there exists a unique line l_{AX} such that $A, X \in l_{AX}$.

We also note that if X and Y are distinct point, then $l_{AX} \neq l_{AY}$. Otherwise, $X, Y \in l_{AX} = l_{AY}$ which forces that $l_{AX} = l_{AY} = l$ which contradicts to the fact that $A \notin l$.

Therefore, there are infinitely many lines passing through A.

(6 points)